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An identity involving Narayana numbers

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Abstract

We present a bijection for a labelled plane forest by using a bijective algorithm for labelled plane trees. This turns out to be a combinatorial explanation for an identity involving Narayana numbers.

1. Introduction

The *Narayana numbers* ([9, Sequence A001263] and [10, Exercise 6.36]) are defined by

$$N_{n,k} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1},$$

where $1 \le k \le n$. The Narayana numbers (in fact, a q-analogue of them) were first studied by MacMahon [4, Article 495] and were later rediscovered by Narayana [5]. It turns out that many statistics of combinatorial structures have the Narayana distribution, see [1,3,6–8,11].

The *Star of David rule* [13] is given by

$$\binom{n}{k}\binom{n+1}{k-1}\binom{n+2}{k+1} = \binom{n}{k-1}\binom{n+1}{k+1}\binom{n+2}{k},$$

for any k and n, which implies that

$$\binom{n}{k+1}\binom{n+1}{k}\binom{n+2}{k+2} = \binom{n}{k}\binom{n+1}{k+2}\binom{n+2}{k+1}.$$

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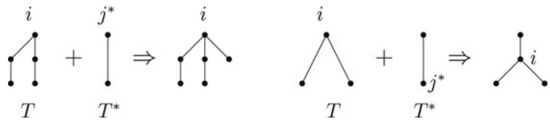

Fig. 1. A horizontal merge and vertical merge.

Multiplying the above two identities and dividing by $n(n+1)(n+2)$ we arrive at

$$N_{n,k+1}N_{n+1,k}N_{n+2,k+2} = N_{n,k}N_{n+1,k+2}N_{n+2,k+1}.$$

In the summer of 2006, Sun [12] asked for a combinatorial proof of the above Narayana identity to be found.

Meanwhile, we find that the product of the bold-faced numbers equals the product of the underlined numbers in the following Narayana matrix, which is a lower triangular matrix with entries $N_{n,k}$:

$$\begin{bmatrix} 1 & & & & & & & \\ 1 & 1 & & & & & & \\ 1 & 3 & 1 & & & & & \\ \underline{1} & 6 & 6 & \mathbf{1} & & & & \\ 1 & 10 & \mathbf{20} & 10 & \underline{1} & & & \\ 1 & \mathbf{15} & 50 & \underline{50} & 15 & 1 & & \\ \mathbf{1} & 21 & \underline{105} & 175 & 105 & 21 & 1 & \\ 1 & \underline{28} & 196 & 490 & \mathbf{490} & 196 & 28 & 1 \end{bmatrix}. \tag{1.1}$$

In this paper, we present a complete answer for these questions by proving a much more general combinatorial identity bijectively. In order to do that, first we introduce *Chen's bijective algorithm* [1] for labelled plane trees and the related involution [2].

In [1], Chen gave a bijective algorithm for decomposing a labelled plane tree on

$$\{1, 2, \ldots, n+1\}$$

into a set of n matchings with labels

$$[2n]^* = \{1, \ldots, n, n+1, (n+2)^*, \ldots, (2n)^*\},$$

where a matching is a rooted plane tree with two vertices. We recall here the reverse procedure of the decomposition algorithm. We consider a set of matchings on $[2n]^*$, where a vertex labelled by the symbol $*$ is called a *marked vertex*.

(1) Find the tree T with the smallest root in which no vertex is marked, and let i be the root of T.
(2) Find the tree T^* that contains the smallest marked vertex j^*.
(3) If j^* is the root of T^*, then merge T and T^* by identifying i and j^*; then keep i as the new vertex and put the subtrees of T^* to the right of T. This operation is called a *horizontal merge*.
    If j^* is a leaf of T^*, then replace j^* with T in T^*. This operation is called a *vertical merge*. Both operations are represented in Fig. 1.
(4) Repeat the above procedure until a desired labelled plane tree is left.

For any set of n matchings labelled by $[2n]^*$, a matching is said to be *mixed* if it consists of an unmarked vertex and a marked vertex.

We now introduce an involution [2] for the labelled plane trees. The involution is built on the set of labelled plane trees whose matching decompositions contain a matching with mixed vertices. Given such a plane tree T, we decompose it into matchings. Then we choose the matching with mixed vertices such that the unmarked vertex is minimum. The involution simply turns the chosen matching upside down. Note that the number of leaves of a plane tree T equals the number of unmarked leaves of the matchings in the corresponding decomposition (see [1]).

In this paper, using Chen's bijective algorithm for labelled plane trees and the related involution, we prove combinatorially

$$\begin{aligned}&N_{n,k+m-1}N_{n+1,k+m-2}N_{n+2,k+m-3}\cdots N_{n+m-2,k+1}N_{n+m-1,k}N_{n+m,k+m}\\&\quad = N_{n,k}N_{n+1,k+m}N_{n+2,k+m-1}\cdots N_{n+m-2,k+3}N_{n+m-1,k+2}N_{n+m,k+1}.\end{aligned}$$

The above identity for $m = 2$ provides what Sun requested, as described above. For $n = m = 4$ and $k = 1$, we have the product in the Narayana matrix (1.1).

2. Proof

It is well known [1,8,10] that the number of plane trees with n edges and k leaves is given by the Narayana number $N_{n,k}$. Note that the number of labelled plane trees with n edges, or $n + 1$ vertices, and k leaves equals $k!(n + 1 - k)!\binom{n+1}{k}$ times the number of unlabelled plane trees with n edges and k leaves, which can be denoted by

$$M_{n,k} = k!(n + 1 - k)!\binom{n+1}{k} N_{n,k} = (n + 1)!N_{n,k}.$$

We define a *plane forest* (resp. *labelled plane forest*) to be a class of plane trees (resp. labelled plane trees) which are linearly ordered. Consider the following labelled plane forest with any integer $1 \le m \le n - k$:

$$(T_{n,k+m-1}; T_{n+1,k+m-2}; T_{n+2,k+m-3}; \cdots T_{n+m-2,k+1}; T_{n+m-1,k}; T_{n+m,k+m}),$$

where $T_{i,j}$ is a labelled plane tree with i edges and j leaves which is enumerated by $M_{i,j}$. We decompose the forest into ordered sets of matchings. For the matchings of $T_{n,k+m-1}$ (resp. $T_{n+m,k+m}$), we choose the matching with mixed vertices such that the unmarked vertex is a leaf and is minimum. Then we turn the chosen matching upside down. We repeat the above procedure inductively $m-1$ times. Thus, the corresponding plane tree $T_{n,k}$ (resp. $T_{n+m,k+1}$) has $m-1$ leaves fewer. For the matchings of $T_{n+\ell,k+m-\ell-1}$ with $1 \le \ell \le m - 1$, we choose the matching with mixed vertices such that the unmarked vertex is a root and is minimum. Then we turn the chosen matching upside down. We repeat this procedure one more time. Thus, the corresponding plane tree $T_{n+\ell,k+m-\ell+1}$ has two leaves more. Hence, the given labelled plane forest corresponds to the following one:

$$(T_{n,k}; T_{n+1,k+m}; T_{n+2,k+m-1}; \cdots T_{n+m-2,k+3}; T_{n+m-1,k+2}; T_{n+m,k+1}).$$

It is easy to see that the above procedure is reversible. Fig. 2 is an illustration of the bijection with $n = m = 2$ and $k = 1$.

So, in this way, we construct a bijection and then we get

$$\begin{aligned}&M_{n,k+m-1}M_{n+1,k+m-2}M_{n+2,k+m-3}\cdots M_{n+m-2,k+1}M_{n+m-1,k}M_{n+m,k+m}\\&\quad = M_{n,k}M_{n+1,k+m}M_{n+2,k+m-1}\cdots M_{n+m-2,k+3}M_{n+m-1,k+2}M_{n+m,k+1}.\end{aligned}$$

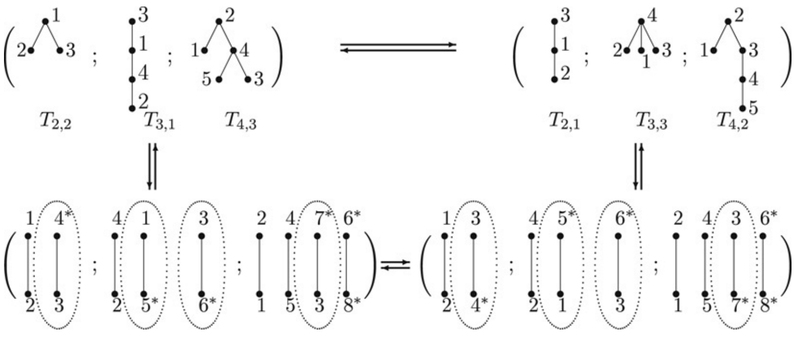

Fig. 2. A bijection on a labelled plane forest.

A straightforward computation gives the following property.

Theorem 2.1. *For all $m \geq 0$,*

$$N_{n,k+m-1}N_{n+1,k+m-2}N_{n+2,k+m-3}\cdots N_{n+m-2,k+1}N_{n+m-1,k}N_{n+m,k+m}$$
$$= N_{n,k}N_{n+1,k+m}N_{n+2,k+m-1}\cdots N_{n+m-2,k+3}N_{n+m-1,k+2}N_{n+m,k+1}.$$

For $m = 1$, it is trivial. For $m = 2$, we have the following identity:

$$N_{n,k+1}N_{n+1,k}N_{n+2,k+2} = N_{n,k}N_{n+1,k+2}N_{n+2,k+1}.$$

For $n = m = 4$ and $k = 1$, we have the product in the Narayana matrix (1.1).

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